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Modelling collective opinion formation by means of active Brownian particles

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Abstract. The concept of active Brownian particles is used to model a collective opinion formation process. It is assumed that individuals in community create a two-component communication field that influences the change of opinions of other persons and/or can induce their migration. The communication field is described by a reaction-diffusion equation, the opinion change of the individuals is given by a master equation, while the migration is described by a set of Langevin equations, coupled by the communication field. In the mean-field limit holding for fast communication we derive a critical population size, above which the community separates into a majority and a minority with opposite opinions. The existence of external support (*e.g.* from mass media) changes the ratio between minority and majority, until above a critical external support the supported subpopulation exists always as a majority. Spatial effects lead to two critical "social" temperatures, between which the community exists in a metastable state, thus fluctuations below a certain critical wave number may result in a spatial opinion separation. The range of metastability is particularly determined by a parameter characterizing the individual response to the communication field. In our discussion, we draw analogies to phase transitions in physical systems.

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1 Introduction

In recent years, there has been a lot of interest in applications of physical paradigms to a *quantitative* description of social [1–7] and economic processes [8–11]

Methods of synergetics [12,13], stochastic processes [14,15], deterministic chaos [16–19] and lattice gas models [20–22] have been successfully applied for this purpose.

The formation of public opinion [23–29] is among the challenging problems in social science, because it reveals a complex dynamics, which may depend on different internal and external influences. We mention the influence of political leaders, the biasing effect of mass media, as well as individual features, such as persuasion or support for other opinions.

A quantitative approach to the dynamics of opinion formation is given by the concept of *social impact* [20,23], which is based on methods similar to the cellular automata approach [30,22]. The social impact describes the force on an individual to keep or to change its current opinion. A short outline of this model is given in Section 2. The equilibrium statistical mechanics of the social impact model was formulated in [20], while in [21,27,29] the occurrence of phase transitions and bistability in the presence of a strong leader or an external impact have been analysed.

Despite these extensive studies of the social impact model, there are several basic disadvantages of the concept. In particular, the social impact theory assumes, that the impact on an individual is updated with infinite velocity, and no memory effects are considered. Further, there is no migration of the individuals, and any "spatial" distribution of opinions refer to a "social", but not to the physical space.

In fact, the model of social impact has not been developed to describe processes of opinion diffusion and migration. In this paper, we present an alternative approach to the social impact model of collective opinion formation, which tries to include these features. Our model is based on *active Brownian particles*, which interact *via* a communication field. This field considers the spatial distribution of the individual opinions, further, it has a certain life time, reflecting a collective memory effect and it can spread out in the community, modeling the transfer of information.

Active Brownian particles [31–33] are Brownian particles with the ability to take up energy from the environment, to store it in an internal depot [34,35] and to

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convert internal energy to perform different activities, such as metabolism, motion, change of the environment, or signal-response behavior. As a specific action, the active Brownian particles (or active walkers, within a discrete approximation) are able to generate a self-consistent field, which in turn influences their further movement and physical or chemical behavior. This non-linear feedback between the particles and the field generated by themselves results in an interactive structure formation process on the macroscopic level. Hence, these models have been used to simulate a broad variety of pattern formation processes in complex systems, ranging from physical to biological and social systems [36–38, 40, 41]

In Section 2, we specify the model of active Brownian particles for the formation of collective opinion structures. In Section 3, we discuss the limiting case of fast communication between the individuals. Further, we investigate the influence of an external support and derive critical parameters for the existence of subpopulations as majorities or minorities. In Section 4, we investigate spatial opinion structures, and estimate critical wave numbers for the fluctuations, which lead to a spatial separation of the opinions. By deriving two different critical temperatures, we draw an analogy to the theory of phase transitions.

2 Stochastic model of opinion change and migration

Let us consider a 2-dimensional spatial system with the total area A, where a community of N individuals (members of a social group) exists. Each of them can share one of two opposite opinions on a given subject, denoted as $\theta_i = \pm 1; i = 1, ..., N$. Here, θ_i is considered as an individual parameter, representing an *internal degree of freedom.* Within a stochastic approach, the probability $p_i(\theta_i, t)$ to find the individual i with the opinion θ_i, changes in the course of time due to the following master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i(\theta_i, t) = \sum_{\theta_i'} w(\theta_i|\theta_i') p_i(\theta_i', t) - p_i(\theta_i, t) \sum_{\theta_i'} w(\theta_i'|\theta_i). \tag{1}$$

Here, $w(\theta_i'|\theta_i)$ means the transition rate to change the opinion θ_i into one of the possible opinions θ_i' during the next time step, with $w(\theta_i|\theta_i) = 0$. In the considered case, there are only two possibilities, either $\theta_i = +1 \to \theta_i' = -1$, or $\theta_i = -1 \to \theta_i' = +1$. In the social impact theory [20, 23], it is assumed that the change of opinions depends on the social impact, I_i, and a "social temperature", T [21, 27]. A possible ansatz for the transition rate reads:

$$w(\theta_i'|\theta_i) = \eta \exp\{I_i/T\}. \tag{2}$$

Here, η [1/s] defines the time scale of the transitions. T represents the erratic circumstances of the opinion change: in the limit $T \to 0$ the opinion change is more determined by I_i, leading to deterministic transitions. As equation (2) indicates, the likelyhood for changing the opinion is rather small, if $I_i < 0$. Hence, a negative social impact on individual i represents a condition for *stability.* To be specific, in the social impact theory, I_i may consist of three parts:

$$I_i = I_i^{\mathrm{p}} + I_i^{\mathrm{s}} + I_i^{\mathrm{ex}} \tag{3}$$

I_i^{p} represents influences imposed on the individual by other members of the group, *e.g.* to change or to keep its opinion. I_i^{s}, on the other hand, is kind of a self-support for the own opinion, $I_i^{\mathrm{s}} < 0$, and I_i^{ex} represents external influences, *e.g.* from government policy, mass media, etc. which may also support a certain opinion.

Within a simplified approach of the social impact theory, every individual can be ascribed a single parameter, the "strength", s_i. Furthermore, a social distance d_{ij} is defined, which measures the distance between each two individuals (i, j) in a *social space* [20, 23], which does not necessarily coincide with the physical space. It is assumed that the impact between two individuals decreases with the social distance in a non-linear manner. The above assumptions are included in the following ansatz [21, 27]:

$$I_i = -\theta_i \sum_{j=1, j\neq i}^{N} s_j \theta_j / d_{ij}^n \; - \varepsilon s_i \; + e_i \theta_i \tag{4}$$

ε is the so-called self-support parameter, and $n > 0$ is a model constant. The external influence, e_i may be regarded as a global preference towards one of the opinions. A negative social impact on individual i is obtained, (i) if most of the opinions in its social vicinity match its own opinion, or (ii) if the impact resulting from opposite opinions is at least not large enough to compensate its self-support, or (iii) if the external influences do not force the individual to change its opinion, regardless of self-support or the impact of the community.

In the form outlined above, the concept of social impact has certain drawbacks: The social impact theory assumes that the impact on an individual is *instantaneously* updated, if some opinions are changed in the group (which basically means a communication with infinite velocity). Spatial effects in a physical space are not considered here, any "spatial" distribution of opinions refers to the social space. Moreover, the individuals are not allowed to move. Finally, no memory effects are considered in the social impact, the community is only affected by the current state of the opinion distribution, regardless of its history and past experience.

In this paper, we want to modify the theory by including some important features of social systems: (i) the existence of a *memory*, which reflects the past experience, (ii) an *exchange of information* in the community with a *finite* velocity, (iii) the influence of *spatial distances* between individuals, (iv) the possibility of *spatial migration* for the individuals. It seems more realistic to us that individuals have the chance to migrate to places where their opinion is supported rather than change their opinion. And in most cases, individuals are not instantaneously affected by the opinions of others, especially if they are not in their close vicinity.

As a basic element of our theory, a scalar *spatio-temporal communication field* $h_\theta(\mathbf{r},t)$ is used. Every individual contributes permanently to this field with its opinion θ_i and with its personal strength s_i at its current spatial location $\mathbf{r}_i$. The information generated this way has a certain life time $1/\beta$ [s], further it can spread throughout the system by a diffusion-like process, where D_h [m^2/s] represents the diffusion constant for information exchange. We have to take into account that there are two different opinions in the system, hence the communication field should also consist of two components, $\theta = \{-1,+1\}$, each representing one opinion. For simplicity, it is assumed that the information resulting from the different opinions has the same life time and the same way of spatial distribution; more complex cases can be considered as well.

The spatio-temporal change of the communication field can be summarized in the following equation:

$$\frac{\partial}{\partial t} h_\theta(\mathbf{r},t) = \sum_{i=1}^{N} s_i\ \delta_{\theta,\theta_i}\ \delta(\mathbf{r}-\mathbf{r}_i) - \beta h_\theta(\mathbf{r},t)\ +\ D_h \Delta h_\theta(\mathbf{r},t). \tag{5}$$

Here, δ_{θ,θ_i} is the Kronecker Delta indicating that the individuals contribute only to the field component which matches their opinion θ_i. $\delta(\mathbf{r}-\mathbf{r}_i)$ means Dirac's Delta function used for continuous variables, which indicates that the individuals contribute to the field only at their current position, $\mathbf{r}_i$. We note that this equation is a stochastic partial differential equation with

$$n^{\text{micr}}(\mathbf{r},t) = \sum_{i=1}^{N} \delta(\mathbf{r}-\mathbf{r}_i) \tag{6}$$

being the microscopic density [32] of the individuals changing their position due to equation (8). Hence, the changes of the communication field $h_\theta(\mathbf{r},t)$ are measured in units of a *density* of the personal strength s_i.

Instead of a social impact, the communication field $h_\theta(\mathbf{r},t)$ influences the individual i as follows: At a certain location $\mathbf{r}_i$, the individual with opinion $\theta_i = +1$ is affected by two kinds of information: the information resulting from individuals who share his/her opinion, $h_{\theta=+1}(\mathbf{r}_i,t)$, and the information resulting from the opponents $h_{\theta=-1}(\mathbf{r}_i,t)$. The diffusion constant D_h determines how fast he/she will receive any information, and the decay rate β determines, how long a generated information will exist. Dependent on the *local* information, the individual has two opportunities to act: (i) it can *change its opinion*, (ii) it can *migrate* towards locations which provide a larger support of its current opinion. These opportunities are specified in the following.

For the change of opinions, we can adopt the transition probability, equation (2), by replacing the influence of the social impact I_i with the influence of the local communication field. A possible ansatz reads:

$$w(\theta_i'|\theta_i) = \eta \exp\{[h_{\theta'}(\mathbf{r}_i,t) - h_\theta(\mathbf{r}_i,t)]/T\}$$
$$w(\theta_i|\theta_i) = 0. \tag{7}$$

As in equation (2), the probability to change opinion θ_i is rather small, if the local field $h_\theta(\mathbf{r}_i,t)$, which is related to the support of opinion θ_i, overcomes the local influence of the opposite opinion. This effect, however, is scaled again by the *social temperature* T, which is a measure for the randomness in social interaction. Note, that the social temperature is measured in units of the communication field.

The movement of the individual located at space coordinate $\mathbf{r}_i$ may depend both on erratic circumstances and on the influence of the communication field. Within a stochastic approach, this movement can be described by the following overdamped Langevin equation:

$$\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \alpha_i\ \frac{\partial h_{\mathrm{e}}(\mathbf{r},t)}{\partial r}\bigg|_{\mathbf{r}_i} + \sqrt{2\,D_n}\ \xi(t). \tag{8}$$

In the last term of equation (8) D_n means the spatial diffusion coefficient of the individuals. The random influences on the movement are modeled by a stochastic force with a δ-correlated time dependence, *i.e.* $\xi(t)$ is the white noise with $\langle \xi_i(t)\, \xi_j(t')\rangle = \delta_{ij}\,\delta(t-t')$.

The term $h_{\mathrm{e}}(\mathbf{r},t)$ in equation (8) means an *effective* communication field which results from $h_\theta(\mathbf{r},t)$ as specified below. It follows that the overdamped Langevin equation (8) considers the response of the individual to the *gradient* of the field $h_{\mathrm{e}}(\mathbf{r},t)$, where α_i is the individual response parameter, weighting the importance of the information received. In the considered case, the effective communication field $h_{\mathrm{e}}(\mathbf{r},t)$ is a certain function of both components, $h_{\pm1}(\mathbf{r},t)$, of the communication field, see equation (5). One can consider different types of response, for example the following:

(i) The individuals try to move towards locations which provide the most support for their current opinion θ_i. In this case, they only count on the information which matches their opinion, $h_{\mathrm{e}}(\mathbf{r},t) = h_\theta(\mathbf{r},t)$, and follow the local ascent of the field ($\alpha_i > 0$).
(ii) The individuals try to move away from locations which provide any negative pressure on their current opinion θ_i. In this case, they count on the information resulting from opposite opinions (θ'), $h_{\mathrm{e}}(\mathbf{r},t) = h_{\theta'}(\mathbf{r},t)$, and follow the local descent of the field ($\alpha_i < 0$).
(iii) The individuals try to move away from locations, if they are forced to change their current opinion θ_i, but they can accept a vicinity of opposite opinions, as long as these are not dominating. In this case, they count on the information resulting from both supporting and opposite opinions, and the local difference between them is important: $h_{\mathrm{e}}(\mathbf{r},t) = [h_\theta(\mathbf{r},t) - h_{\theta'}(\mathbf{r},t)]$ with $\alpha_i > 0$.

Additionally, the response parameter can also consider that the response occurs only, if the absolute value of the effective field is locally above a certain threshold h_{thr}: $\alpha_i = \Theta\left[|h_{\mathrm{e}}(\mathbf{r},t)| - h_{\text{thr}}\right]$, with $\Theta[y]$ being the Heavyside function: $\Theta = 1$, if $y > 0$, otherwise $\Theta = 0$. We note that for the further discussions in Sections 3 and 4, we assume $h_{\mathrm{e}}(\mathbf{r},t) = h_\theta(\mathbf{r},t)$ for the effective communication

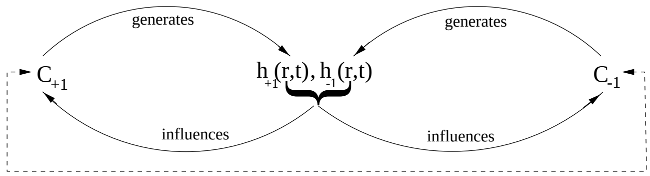


Fig. 1. Circular causation between the individuals with different opinions, C_{-1}, C_{+1} and the two-component communication field, $h_\theta(\mathbf{r}, t)$.

field (case i), while $\alpha_i \equiv \alpha$ is treated as a positive constant independent of i and $h_\mathrm{e}(\mathbf{r}, t)$.

In order to summarize our model, we note the nonlinear feedback between the individuals and the communication field as shown in Figure 1. The individuals generate the field, which in turn influences their further movement and their opinion change. In terms of synergetics, the field plays the role of an order parameter, which couples the individual actions, and this way initiates spatial structures and coherent behavior within the social group.

The complete dynamics of the community can be formulated in terms of the canonical N-particle distribution function

$$P(\underline{\theta}, \underline{r}, t) = P(\theta_1, \mathbf{r}_1, ..., \theta_N, \mathbf{r}_N, t), \tag{9}$$

which gives the probability to find the N individuals with the opinions $\theta_1, ..., \theta_N$ in the vicinity of $\mathbf{r}_1, ..., \mathbf{r}_N$ on the surface A at time t. Considering both opinion changes and movement of the individuals, the master equation for $P(\underline{\theta}, \underline{r}, t)$ reads:

$$\begin{aligned} \frac{\partial}{\partial t} P(\underline{\theta}, \underline{r}, t) = & \sum_{\underline{\theta}' \neq \underline{\theta}} \Big[w(\underline{\theta}|\underline{\theta}') P(\underline{\theta}', \underline{r}, t) - w(\underline{\theta}'|\underline{\theta}) P(\underline{\theta}, \underline{r}, t) \Big] \\ & - \sum_{i=1}^{N} \Big[\nabla_i \, (\alpha \, \nabla_i h_\theta(\mathbf{r}, t) \, P(\underline{\theta}, \underline{r}, t)) - D_n \, \Delta_i P(\underline{r}, \underline{\theta}, t) \Big]. \end{aligned} \tag{10}$$

The first line of the right-hand side of equation (10) describes the "gain" and "loss" of individuals (with the coordinates $\mathbf{r}_1, ..., \mathbf{r}_N$) due to opinion changes, where $w(\underline{\theta}|\underline{\theta}')$ means any possible transition within the opinion distribution $\underline{\theta}'$ which leads to the assumed distribution $\underline{\theta}$. The second line describes the change of the probability density due to the motion of the individuals on the surface. Equation (10) together with equations (5, 7) forms a complete description of our system.

3 The case of fast communication

3.1 Derivation of mean value equations

Let us first restrict to the case of very fast exchange of information in the system. Then, spatial inhomogenities are equalized immediately, hence, the communication field $h_\theta(\mathbf{r}, t)$ can be approximated by a mean field $\bar{h}_\theta(t)$:

$$\bar{h}_\theta(t) = \frac{1}{A} \int_A h_\theta(\mathbf{r}, t) \, \mathrm{d}r^2, \tag{11}$$

where A means the system size. The equation for the mean field $\bar{h}_\theta(t)$ results from equation (5):

$$\frac{\partial \bar{h}_\theta(t)}{\partial t} = -\beta \bar{h}_\theta(t) + s \bar{n}_\theta \tag{12}$$

with $s_i \equiv s$ and the mean density

$$\bar{n}_\theta = \frac{N_\theta}{A} \; ; \quad \bar{n} = \frac{N}{A}, \tag{13}$$

where the number of individuals with a given opinion θ fulfils the condition

$$\sum_\theta N_\theta = N_{+1} + N_{-1} = N = \mathrm{const}. \tag{14}$$

We note that in the mean–field approximation no spatial gradients in the communication field exist. Hence, there is no additional driving force for the individuals to move, as assumed in equation (8). Such a situation can be imagined for communities existing in very small systems with small distances between different groups. In particular, in such small communities also the assumption of a fast information exchange holds. Thus, in this section, we restrict our discussion to subpopulations with a certain opinion rather than to individuals at particular locations.

Let $p(N_\theta, t)$ denote the probability to find N_θ individuals in the community which shares opinion θ. The master equation for $p(N_{+1}, t)$ explicitely reads:

$$\begin{aligned} \frac{\partial}{\partial t} p(N_{+1}, t) = & \; W(N_{+1}|N_{+1} - 1) \, p(N_{+1} - 1, t) \\ & + W(N_{+1}|N_{+1} + 1) \, p(N_{+1} + 1, t) \\ & - p(N_{+1}, t) \, [W(N_{+1} + 1|N_{+1}) \\ & + W(N_{+1} - 1|N_{+1})] \, . \end{aligned} \tag{15}$$

The transition rates $W(M|N)$ appearing in equation (15) are assumed to be proportional to the probability to change a given opinion, equation (7), and to the number of individuals which can change their opinion into the given direction:

$$\begin{aligned} W(N_{+1} + 1|N_{+1}) &= N_{-1} \, \eta \, \exp\{(\bar{h}_{+1} - \bar{h}_{-1})/T\}, \\ W(N_{+1} - 1|N_{+1}) &= N_{+1} \, \eta \, \exp\{-(\bar{h}_{+1} - \bar{h}_{-1})/T\}. \end{aligned} \tag{16}$$

The mean values for the number of individuals with a certain opinion can be derived from the master equation (15)

$$\langle N_\theta(t) \rangle = \sum_{\{N_\theta\}} N_\theta \; p(N_\theta, t), \tag{17}$$

where the summation is over all possible numbers of N_θ which obey the condition equation (14). From equation (17), the deterministic equation for the change of $\langle N_\theta \rangle$

can be derived in the first approximation as follows [42] (see also [1,3,7]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle N_\theta\rangle = \langle W(N_\theta+1|N_\theta) - W(N_\theta-1|N_\theta)\rangle \cdot \tag{18}$$

For N_{+1}, this equation reads explicitely:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle N_{+1}\rangle = \Bigg\langle N_{-1}\,\eta\,\exp\left[\frac{\bar{h}_{+1}(t)-\bar{h}_{-1}(t)}{T}\right] - N_{+1}\,\eta\,\exp\left[-\frac{\bar{h}_{+1}(t)-\bar{h}_{-1}(t)}{T}\right]\Bigg\rangle \cdot \tag{19}$$

Introducing now the fraction of a *subpopulation* with opinion θ, $x_\theta = \langle N_\theta\rangle/N$, and using the standard approximation to factorize equation (19), we can write it as:

$$\begin{aligned} \dot{x}_{+1} &= (1-x_{+1})\,\eta\,\exp(a) - x_{+1}\,\eta\,\exp(-a),\\ a &= \left[\bar{h}_{+1}(t)-\bar{h}_{-1}(t)\right]/T. \end{aligned} \tag{20}$$

Via $\Delta\bar{h}(t) = \bar{h}_{+1} - \bar{h}_{-1}$, this equation is coupled with the equation

$$\Delta\dot{\bar{h}} = -\beta\,\Delta\bar{h} + s\bar{n}\Big(2x_{+1}-1\Big) \tag{21}$$

which results from equation (12) for the two field components.

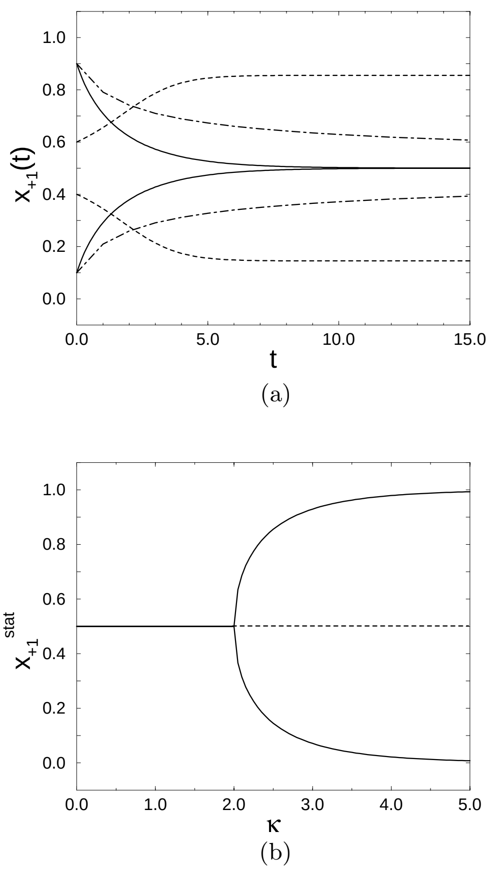


Fig. 2. (a) Time dependence of the fraction $x_{+1}(t)$ (Eq. (20)) of the subpopulation with opinion +1 for different initial conditions and for three different values of κ: 1.0 (solid line); 2.0 (dot-dashed line), 3.0 (dashed line). (b) Stationary solutions for x_{+1} (Eq. (23)) for different values of κ. The bifurcation at the critical value $\kappa^{\mathrm{c}} = 2$ is clearly visible.

3.2 Critical and stable subpopulationm sizes

Within a quasistationary approximation, we can assume that the communication field *relaxes faster* than the distribution of the opinions into a stationary state. Hence, with $\dot{\bar{h}}_\theta = 0$, we find from equation (12):

$$\begin{aligned} \bar{h}_{+1}^{\mathrm{stat}} = \frac{s\,\bar{n}}{\beta}x_{+1} \quad &; \quad \bar{h}_{-1}^{\mathrm{stat}} = \frac{s\,\bar{n}}{\beta}(1-x_{+1})\\ a = \kappa\left(x_{+1}-\frac{1}{2}\right) &\text{ with } \kappa = \frac{2s\,\bar{n}}{\beta T}\cdot \end{aligned} \tag{22}$$

Here, the parameter κ includes the specific *internal conditions* within the community, such as the total population size, the social temperature, the individual strength of the opinions, or the life time of the information generated. Inserting a from equation (22) into equation (22), a closed equation for $\dot{x}_\theta$ is obtained, which can be integrated with respect to time (Figure 2a). We find that, depending on κ, different stationary values for the fraction of the subpopulations exist. For the critical value, $\kappa^{\mathrm{c}} = 2$, the stationary state can be reached only asymptotically. Figure 2b shows the stationary solutions, $\dot{x}_\theta = 0$, resulting from the equation for x_{+1}:

$$(1-x_{+1})\,\exp\left[\kappa\,x_{+1}\right] = x_{+1}\,\exp\left[\kappa\,(1-x_{+1})\right]. \tag{23}$$

For $\kappa < 2$, $x_{+1} = 0.5$ is the only stationary solution, which means a stable community where both opposite opinions have the same influence. However, for $\kappa > 2$, the equal distribution of opinions becomes unstable, and a separation process towards a preferred opinion is obtained, where $x_{\pm 1} = 0.5$ plays the role of a separation line. We find now two stable solutions where both opinions coexist with different shares in the community, as shown in Figure 2. Hence, each subpopulation can exist either as a *majority* or as a *minority* within the community. Which of these two possible situations is realized, depends in a deterministic approach on the initial fraction of the subpopulation. For initial values of x_{+1} below the separatrix, 0.5, the minority status will be most likely the stable situation, as Figure 2a shows.

The bifurcation occurs at $\kappa^{\mathrm{c}} = 2$, where the former stable solution $x_{+1} = 0.5$ becomes unstable. From the condition $\kappa = 2$ we can derive a *critical population size*,

$$N^{\mathrm{c}} = \beta\,A\,T/s, \tag{24}$$

where for larger populations an equal fraction of opposite opinions is certainly unstable. If we consider *e.g.* a *growing community* with fast communication, then both contradicting opinions are balanced, as long as the population number is small. However, for $N > N^{\mathrm{c}}$, *i.e.* after a certain population growth, the community tends towards one of these opinions, thus necessarily separating into a majority and a minority. Which of these opinions would be dominating, depends on small fluctuations in the bifurcation point, and has to be investigated within a stochastic approach. We note that equation (24) for the critical population size can be also interpreted in terms of a critical social temperature, which leads to an opinion separation in the community. This will be discussed in more detail in Section 4.

From Figure 2b, we see further, that the stable coexistence between majority and minority breaks down at a certain value of κ, where almost the whole community shares the same opinion. From equation (23) it is easy to find that *e.g.* $\kappa \approx 4.7$ yields $x_{+1} \approx \{0.01; 0.99\}$, which means that about 99% of the community share either opinion +1 or −1.

3.3 Influence of external support

Now, we discuss the situation that the symmetry between the two opinions is broken due to external influences on the individuals. We may consider two similar cases: (i) the existence of a *strong leader* in the community, who possesses a strength s_l which is much larger than the usual strength s of the other individuals, (ii) the existence of an external field, which may result from government policy, mass media, etc. which support a certain opinion with a strength s_m.

The additional influence $s_{ext} := \{s_l/A, s_m/A\}$ mainly effects the communication field, equation (5), due to an extra contribution, normalized by the system size A.

If we assume an external support of opinion $\theta = +1$, the corresponding field equation in the mean–field limit (Eq. (12)) and the stationary solution (Eq. (22)) are changed as follows:

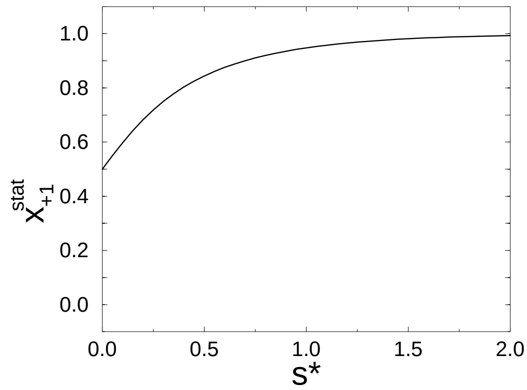
$$\begin{aligned}\dot{\bar{h}}_{+1} &= -\beta \bar{h}_{+1}(t) + s\,\bar{n}\,x_{+1} + s_{ext} \\ \bar{h}_{+1}^{stat} &= \frac{s\,\bar{n}}{\beta} x_{+1} + \frac{s_{ext}}{\beta} \\ a &= \kappa\left(x_{+1} - \frac{1}{2}\right) + \frac{s_{ext}}{\beta T}. \end{aligned} \tag{25}$$

Hence, in equation (23) which determines the stationary solutions, the arguments are shifted by a certain value:

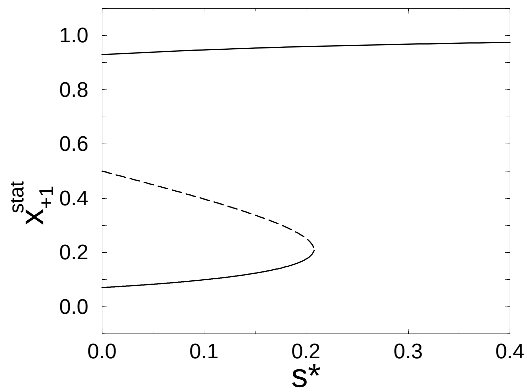
$$\begin{aligned}&(1 - x_{+1})\,\exp\left[\kappa\,x_{+1} + \frac{s_{ext}}{\beta T}\right] \\ &= x_{+1}\,\exp\left[\kappa\,(1 - x_{+1}) - \frac{s_{ext}}{\beta T}\right]. \end{aligned} \tag{26}$$

Figure 3 shows how the critical and stable subpopulation sizes change for subcritical and supercritical values of κ, dependent on the strength of the external support.

For $\kappa < \kappa^c$ (Fig. 3a), we see that there is still only one stable solution, but with an increasing value of s_{ext}, the supported subpopulation exists as a majority. For $\kappa > \kappa^c$ (Fig. 3b), we observe again two possible stable situations for the supported subpopulation, either a minority or a majority status. But, compared to Figure 2b, the symmetry between these possibilities is now broken due to the external support, which increases the region of initial conditions leading to a majority status.



(a)



(b)

Fig. 3. Stable fraction of the subpopulation, x_{+1}^{stat}, as a function of the strength $s^\star = s_{ext}/\beta T$ of the external support. (a) $\kappa = 1$, (b) $\kappa = 3$. The dashed line in (b) represents the separation line for the initial conditions, which lead either to a minority or a majority status of the subpopulation.

Interestingly, at a critical value of s_{ext}, the possibility of a minority status completely vanishes. Hence, for a certain supercritical external support, the supported subpopulation will grow towards a majority, regardless of its initial population size, with no chance for the opposite opinion to be established. This situation is quite often realized in communities with one strong political or religious leader ("fundamentalistic dictatorships"), or in communities driven by external forces, such as financial or military power ("banana republics").

The value of the critical external support, s_{ext}^c, of course depends on κ, which summarizes the internal situation in terms of the social temperature, or the population size, etc. From equation (26) we can derive the condition for which two of the three possible solutions coincide, thus

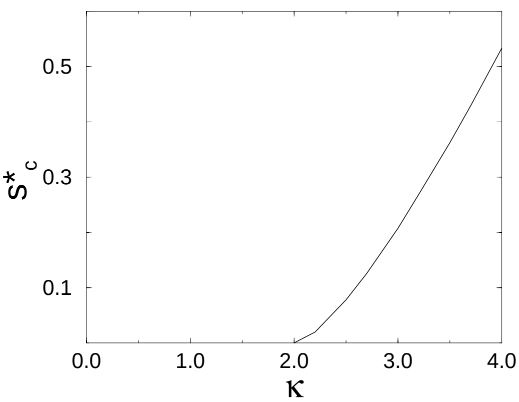


Fig. 4. Critical external support $s_c^\star$ (Eq. (27)) as a function of κ.

determining the relation between $s^{\rm c}_{\rm ext}$ and κ as follows:

$$s^\star_{\rm c} = \frac{s^{\rm c}_{\rm ext}}{\beta T} = \frac{1}{2}\ln\left[\frac{1-\sqrt{1-\frac{2}{\kappa}}}{1+\sqrt{1-\frac{2}{\kappa}}}\right] + \frac{1}{2}\kappa\sqrt{1-\frac{2}{\kappa}}\cdot \qquad (27)$$

Figure 4 shows how much external support is needed to paralyze a community with a given internal situation (κ) by one ruling opinion. As one can see, the critical external support is an increasing function of the parameter κ, meaning that it is more difficult to paralyse a society with strong interpersonal interactions.

Let us conclude the discussion of the phase transition in the mean field limit, presented in this section. With respect to the social impact theory [23,5,20,21,29], we note that phase transitions have been not considered there, since the focus was on other phenomena so far. On the other hand, for the case of two opinions, our results well correspond to those obtained by Weidlich and Haag in a model of collective opinion formation [1,3]. There, a master equation and appropriate *utility potentials* are used to analyse the mean-field dynamics of interacting populations. Similar to our model, the approach used in [1,3] leads to a phase transition and a corresponding bifurcation diagram. The main difference between both models is in the interpretation of the bifurcation parameter κ. In our case, κ results from other model parameters, *e.g.* from the mean "social strength" s which plays a role of a coupling constant between the opinions and the communication field. In the model of Weidlich and Haag, on the other hand, this parameter is interpreted as a derivative of *utility potentials*. Our analytic result for the critical external support, equation (27), is in qualitative agreement with the stability analysis and the computer simulations presented in [1,3].

4 Critical conditions for spatial opinion separation

In the previous section, the existence of critical parameters, such as $\kappa^{\rm c}$ or $s^{\rm c}_{\rm ext}$, has been proven for a community with fast communication, where no inhomogenities in the communication field can exist. In the more realistic case, however, we have finite diffusion coefficients for the information, and the mean-field approximation, equation (12), is no longer valid. Instead of focussing on the subpopulation sizes, we now need to consider the *spatial distribution* of individuals with opposite opinions.

Starting with the canonical N–particle distribution function, $P(\underline{\theta},\underline{r},t)$, equation (10), the spatio-temporal density of individuals with opinion θ can be obtained as follows:

$$n_\theta(\mathbf{r},t) = \int \sum_{i=1}^{N} \delta_{\theta,\theta_i}\, \delta(\mathbf{r}-\mathbf{r}_i) \times P(\theta_1,\mathbf{r}_1...,\theta_N,\mathbf{r}_N,t)\, {\rm d}\mathbf{r}_1...{\rm d}\mathbf{r}_N. \qquad (28)$$

Integrating equation (10) according to equation (28) and neglecting higher order correlations, we obtain the following reaction-diffusion equation for $n_\theta(\mathbf{r},t)$

$$\frac{\partial}{\partial t}\, n_\theta(\mathbf{r},t) = -\,\nabla\Big[n_\theta(\mathbf{r},t)\,\alpha\,\nabla h_\theta(\mathbf{r},t)\Big] + D_n\,\Delta n_\theta(\mathbf{r},t) - \sum_{\underline{\theta'\neq\theta}}\Big[w(\theta'|\theta)\,n_\theta(\mathbf{r},t) + w(\theta|\theta')\,n_{\theta'}(\mathbf{r},t)\Big] \qquad (29)$$

with the transition rates obtained from equation (7):

$$w(\theta'|\theta) = \eta\,\exp\{[h_{\theta'}(\mathbf{r},t) - h_\theta(\mathbf{r},t)]/T\}$$
$$w(\theta|\theta) = 0. \qquad (30)$$

With $\theta=\{+1,-1\}$, equation (29) is a set of two reaction-diffusion equations, coupled both *via* $n_\theta(\mathbf{r},t)$ and $h_\theta(\mathbf{r},t)$. Inserting the densities $n_\theta(\mathbf{r},t)$ and neglecting any external support, equation (5) for the spatial communication field can be transformed into the linear deterministic equation:

$$\frac{\partial}{\partial t} h_\theta(\mathbf{r},t) = s\,n_\theta(\mathbf{r},t) \;-\; \beta\,h_\theta(\mathbf{r},t) \;+\; D_h\Delta h_\theta(\mathbf{r},t). \qquad (31)$$

The solutions for the spatio-temporal distributions of individuals and opinions are now determined by the four coupled equations, equation (29) and equation (31). For our further discussion, we assume again that the spatio-temporal communication field *relaxes faster* than the related distribution of individuals into a quasi-stationary equilibrium. The field $h_\theta(\mathbf{r},t)$ should still depend on time and space coordinates, but, due to the fast relaxation, there is a fixed relation to the spatio-temporal distribution of individuals. Further, we neglect the independent diffusion of information, assuming that the spreading of opinions is due to the migration of the individuals. From equation (31), we find with $\dot{h}_\theta(\mathbf{r},t)=0$ and $D_h=0$:

$$h_\theta(\mathbf{r},t) = \frac{s}{\beta}\,n_\theta(\mathbf{r},t) \qquad (32)$$

which can now be inserted into equation (29), thus reducing the set of coupled equations to two equations.

The homogeneous solution for $n_\theta(\mathbf{r},t)$ is given by the mean densities:

$$\bar{n}_\theta = \langle n_\theta(\mathbf{r},t)\rangle = \frac{\bar{n}}{2}\,\cdot \tag{33}$$

Under certain conditions however, the homogeneous state becomes unstable and a spatial separation of opinions occurs. In order to investigate these critical conditions, we allow small fluctuations around the homogeneous state $\bar{n}_\theta$:

$$n_\theta(\mathbf{r},t) = \bar{n}_\theta + \delta n_\theta\ ; \quad \left|\frac{\delta n_\theta}{\bar{n}_\theta}\right| \ll 1. \tag{34}$$

Inserting equation (34) into equation (29), a linearization gives:

$$\frac{\partial \delta n_\theta}{\partial t} = \left[D_n - \frac{\alpha s\,\bar{n}}{2\beta}\right]\ \Delta\delta n_\theta + \left[\frac{\eta\, s\,\bar{n}}{\beta T} - \eta\right]\ (\delta n_\theta - \delta n_{-\theta}). \tag{35}$$

With the ansatz

$$\delta n_\theta \sim \exp\left(\lambda t + \mathrm{i}\mathbf{k}\mathbf{r}\right) \tag{36}$$

we find from equation (35) the dispersion relation $\lambda(\mathbf{k})$ for small inhomogeneous fluctuations with wave vector $\mathbf{k}$. This relation yields two solutions:

$$\begin{aligned} &\lambda_1(\mathbf{k}) = -k^2\, C + 2B\ ;\ \ \lambda_2(\mathbf{k}) = -k^2\, C \\ &B = \frac{\eta\, s\,\bar{n}}{\beta T} - \eta\ ;\ \ C = D_n - \frac{\alpha s\,\bar{n}}{2\beta}\cdot \end{aligned} \tag{37}$$

For homogeneous fluctuations we obtain from equation (37)

$$\lambda_1 = \frac{2\,\eta\, s\,\bar{n}}{\beta T} - 2\,\eta;\quad \lambda_2 = 0 \quad \text{for } \mathbf{k} = 0 \tag{38}$$

which means that the homogeneous system is marginally stable as long as $\lambda_1 < 0$, or $s\,\bar{n}/\beta T < 1$. This result agrees with the condition $\kappa < 2$ obtained from the previous mean field investigations in Section 3. The condition $\kappa = 2$ or $B = 0$, respectively, defines a *critical social temperature*

$$T_1^{\mathrm{c}} = \frac{s\,\bar{n}}{\beta}\,\cdot \tag{39}$$

For temperatures $T < T_1^{\mathrm{c}}$, the homogeneous state $n_\theta(\mathbf{r},t) = \bar{n}/2$, where individuals of both opinions are equally distributed, becomes unstable and the spatial separation process occurs. This is in direct analogy to the phase transition obtained from the Ising model of a ferromagnet. Here, the state with $\kappa < 2$ or $T > T_1^{\mathrm{c}}$, respectively, corresponds to the *paramagnetic* or disordered phase, while the state with $\kappa > 2$ or $T < T_1^{\mathrm{c}}$, respectively, corresponds to *ferromagnetic* ordered phase.

The conditions of equation (38) denote a *homogeneous* stability condition. To obtain stability against inhomogeneous fluctuations of wave vector $\mathbf{k}$, the two conditions $\lambda_1(\mathbf{k}) \le 0$ and $\lambda_2(\mathbf{k}) \le 0$ have to be satisfied.

Taking into account the critical temperature T_1^{c}, equation (39), we can rewrite these conditions, equation (37), as follows:

$$\begin{aligned} \mathbf{k}^2\left(D_n - D_n^{\mathrm{c}}\right) - 2\,\eta\left(\frac{T_1^{\mathrm{c}}}{T} - 1\right) &\ge 0 \\ \mathbf{k}^2\left(D_n - D_n^{\mathrm{c}}\right) &\ge 0. \end{aligned} \tag{40}$$

Here, a *critical diffusion coefficient* D_n^{c} for the individuals appears, which results from the condition $C = 0$:

$$D_n^{\mathrm{c}} = \frac{\alpha}{2}\,\frac{s\,\bar{n}}{\beta}\,\cdot \tag{41}$$

Hence, the condition

$$D_n > D_n^{\mathrm{c}}. \tag{42}$$

denotes a second stability condition. In order to explain its meaning, let us consider that the diffusion coefficient of the individuals, D_n, may be a function of the social temperature, T. This sounds reasonable since the social temperature is a measure of randomness in social interaction, and an increase of such a randomness leads to an increase of a random spatial migration. The simplest relation for a function $D_n(T)$ is the linear one, $D_n = \mu T$. By assuming this, we may rewrite equation (40) using a *second critical temperature*, T_2^{c} instead of a critical diffusion coefficient D_n^{c}:

$$\begin{aligned} \mathbf{k}^2\,\mu\left(T - T_2^{\mathrm{c}}\right) - 2\,\eta\left(\frac{T_1^{\mathrm{c}}}{T} - 1\right) &\ge 0 \\ \mathbf{k}^2\,\mu\left(T - T_2^{\mathrm{c}}\right) &\ge 0. \end{aligned} \tag{43}$$

The second critical temperature T_2^{c} reads as follows:

$$T_2^{\mathrm{c}} = \frac{\alpha}{2\mu}\,\frac{s\,\bar{n}}{\beta} = \frac{\alpha}{2\mu}\,T_1^{\mathrm{c}}. \tag{44}$$

The occurrence of two critical social temperatures T_1^{c}, T_2^{c} allows a more detailed discussion of the stability conditions. Therefore, we have to consider two separate cases of equation (44): (1) $T_1^{\mathrm{c}} > T_2^{\mathrm{c}}$ and (2) $T_1^{\mathrm{c}} < T_2^{\mathrm{c}}$, which correspond either to the condition $\alpha < 2\mu$, or $\alpha > 2\mu$, respectively.

In the first case, $T_1^{\mathrm{c}} > T_2^{\mathrm{c}}$, we can discuss three ranges of the temperature T:

(i) For $T > T_1^{\mathrm{c}}$ both eigenvalues $\lambda_1(\mathbf{k})$ and $\lambda_2(\mathbf{k})$, equation (37), are nonpositive for all wave vectors $\mathbf{k}$ and the homogeneous solution $\bar{n}/2$ is *completely stable.*

(ii) For $T_1^{\mathrm{c}} > T > T_2^{\mathrm{c}}$ the eigenvalue $\lambda_2(\mathbf{k})$ is still nonpositive for all values of $\mathbf{k}$, but the eigenvalue $\lambda_1(\mathbf{k})$ is negative only for wave vectors that are larger than some critical value $\mathbf{k}^2 > \mathbf{k}_{\mathrm{c}}^2$:

$$\mathbf{k}_{\mathrm{c}}^2 = \frac{2\,\eta}{\mu\, T}\,\frac{T_1^{\mathrm{c}} - T}{T - T_2^{\mathrm{c}}}\,\cdot \tag{45}$$

This means that, in the given range of temperatures, the homogeneous solution $\bar{n}/2$ is *metastable* in an infinite system, because it is stable only against fluctuations with large wave numbers, *i.e.* against small-scale fluctuations. Large-scale fluctuations destroy the homogeneous state and result in a spatial separation process, *i.e.* instead of a homogeneous distribution of opinions, individuals with the same opinion form separated *spatial domains* which coexist. The range of the metastable region is especially determined by the value of $\alpha < 2\mu$, which defines the difference between T_1^c and T_2^c.

(iii) For $T < T_2^c$ both eigenvalues $\lambda_1(\mathbf{k})$ and $\lambda_2(\mathbf{k})$ are positive for all wave vectors $\mathbf{k}$ (except $\mathbf{k} = 0$, for which $\lambda_2 = 0$ yields), which means that the homogeneous solution $\bar{n}/2$ is *completely unstable*. On the other hand all systems with spatial dimension $L < 2\pi/k_c$ are stable in this temperature region.

For case (2), $T_1^c < T_2^c$, which corresponds to $\alpha > 2\mu$, already small inhomogeneous fluctuations result in an instability of the homogeneous state for $T < T_2^c$, *i.e.* we have a direct transition from the completely stable to the completely unstable regime at the critical temperature $T = T_2^c$.

That means the second critical temperature T_2^c marks the transition into complete instability. The metastable region, which exists for $\alpha < 2\mu$, is bound by the two critical social temperatures, T_1^c and T_2^c. This allows us again to draw an analogy to the theory of phase transitions [43]. It is well known from phase diagrams that the density-dependent *coexistence* curve $T_1^c(\bar{n})$ divides stable and metastable regions, therefore we can name the critical temperature T_1^c, equation (39), as the *coexistence* temperature, which marks the transition into the metastable regime. On the other hand, the metastable region is separated from the completely unstable region by a second curve $T_2^c(\bar{n})$, known as the spinodal curve, which defines the region of *spinodal decomposition*. Hence, we can identify the second critical temperature T_2^c, equation (44), as the *instability* temperature.

We note that similar investigations of the critical system behavior can be performed by discussing the dependence of the stability conditions on the "social strength" s or on the total population number $N = A\bar{n}$. These investigations allow the calculation of a phase diagram for the opinion change in the model discussed, where we can derive critical *population densities* for the spatial opinion separation within the community.

5 Conclusions

We have discussed a simple model of collective opinion formation, based on active Brownian particles, which represent the individuals. Every individual shares one of two opposite opinions and indirectly interacts with its neighbours due to a communication field, which contains the information about the spatial distribution of the different opinions. This two-component field has a certain lifetime, which models memory effects. Furthermore, it can spread out in the community, which describes the diffusion of information. This way, every individual locally receives information about the opinion distribution, which affects its further actions: (i) the individual can keep or change its current opinion, or (ii) it can stay or migrate towards regions where its current opinion is supported. Both actions depend (a) on a social temperature, which describes the stochastic influences, and (b) on the local strength of the communication field, which expresses the deterministic influences of the decision of an individual.

For supercritical conditions within the community (*e.g.* supercritical population size, or supercritical external pressure, or low temperature etc.), the non-linear feedback between the individuals and the communication field, created by themselves, results in a process of *spatial opinion separation*. In this case, the individuals either change their opinion to match the conditions in their neighbourhood, or they keep their opinion, but migrate into regions which support this opinion.

In this paper, we have studied the critical conditions, which may lead to this separation process. In the spatially homogeneous case, which holds either for small communities or for an information exchange with infinite velocity, the communication field can be described in a mean-field approximation.

For this case, we derived a critical population size, N^c (which is related to a critical social temperature, T_1^c). For $N < N^c$, there is a stable balance where both opinions are shared by an equal number of individuals. For $N > N^c$, however, one of these opinions becomes preferred, hence, majorities and minorities appear in the community. Further, we have shown how these majorities change if we consider an external support for one of the opinions. We found, that beyond some critical support, the supported subpopulation must always exist as a majority, since the possibility of its minority status simply disappears.

As a second case, we have investigated a spatially inhomogeneous communication field, which is locally coupled to the distribution of the individuals. This coupling is due to an adiabatically fast relaxation of the communication field into a quasistationary equilibrium.

Using this adiabatic approximation, we were able to derive critical conditions for a *spatial* separation of opinions. We found that above the critical population size (or for $T < T_1^c$), the community could be described as a metastable system, which expresses stability against small-scale perturbations. The region of metastability is bound by a second critical temperature, T_2^c, which describes the transition into instability, where every perturbation results in an immediate separation. Further, we obtained that the range of metastability is particularly determined by the parameter α, which characterizes how strong an individual responses to the information received from the communication field.

Finally, we would like to note that our model of collective opinion formation only sketches some basic features of structure formation in social systems. There is no doubt, that in real human societies a more complex behavior among the individuals occurs, and that decision making and opinion formation may depend on numerous influences beyond a quantitative description. In this paper, we restricted ourselves to a simplified dynamical approach, which purposely stretches some analogies between physical and social systems. The results, however, display similarities to phenomena observed in social systems and allow an interpretation within such a context. So, our model may give rise to further investigations in the field of quantitative sociology.

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